logical rules [tautology [T] vs contradiction [F]]

1. De morgan's law $\begin{cases} \neg(p \wedge q) = \neg p \vee \neg q \\ \neg(p \vee q) = \neg p \wedge \neg q \end{cases}$

2. contrapositive

$$\underbrace{P \to Q \equiv \neg P \vee Q}_{\text{conditional-disjunction}} \equiv Q \vee \neg P \equiv \neg Q \Rightarrow \neg P$$

3. R if S $\Leftrightarrow$ if S then R $\Leftrightarrow$ S implies R

· Set theory

1. * $A - B = \{x \in U \mid x \in A \wedge x \notin B\}$ $\quad |A-B| = |A| - |A \cap B|$

* $|P(S)| = 2^{|S|}$ (power set)

* Set partition: non-empty sets $\{A_1, \cdots A_n\}$ is a partition (划分) of A iff:

① $A = \bigcup_{i=1}^{n} A_i$ ② $A_1, \cdots A_n$ are pairwise disjoint

* Cartesian product $A \times B = \{(x,y) \mid x \in A \wedge y \in B\}$

2. $(A \cap B) \times C = (A \times C) \cap (B \times C)$

proof: let (x, y) be an arbitrary element

$(x,y) \in (A \cap B) \times C$ iff. $x \in A \cap B$, $y \in C$

$(x,y) \in (A \times C) \cap (B \times C)$ iff. $(x,y) \in A \times C$ and $(x,y) \in B \times C$

i.e. $x \in A$ and $x \in B$, $y \in C$ $\Rightarrow x \in A \cap B$. $y \in C$

3. $\begin{cases} \text{complement law} & A \cup A^c = U \quad A \cap A^c = \emptyset \\ \text{De Morgan's law} & (A \cup B)^c = A^c \cap B^c \quad (A \cap B)^c = A^c \cup B^c \\ \text{set difference law} & A - B = A \cap B^c \end{cases}$

· First-order logic

1. quantifiers (양화) $\forall$ & $\exists$

A quantified argument is valid if the conclusion is true whatever the assumptions are true.

2. $\begin{cases} \text{universal instantiation 全称示例} & \forall x.P(x) \quad \therefore P(a) \\ \text{universal modus ponens 全称肯定前件} & \forall x.P(x)\to Q(x) \quad P(a) \therefore Q(a) \\ \text{universal modus tollens 全称否定后件} & \forall x.P(x)\to Q(x) \quad \neg Q(a) \therefore \neg P(a) \end{cases}$

· Mathematical Induction

[e.g.] $\forall n \ge 2 \quad \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} > \sqrt{n}$

① when $n=2 \quad \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} = \frac{\sqrt{2}+2}{2} > \sqrt{2} = \frac{2\sqrt{2}}{2}$

② $P(n) \Rightarrow P(n+1) \quad \frac{1}{\sqrt{1}} + \cdots + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1}} > \sqrt{n} + \frac{1}{\sqrt{n+1}}$

prove $\sqrt{n} + \frac{1}{\sqrt{n+1}} > \sqrt{n+1} \iff \sqrt{n} > \frac{n}{\sqrt{n+1}} \iff n(n+1) > n^2$

[e.g.2] $\left(\sum_{i=1}^{n} a_i b_i\right)^2 \le \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right)$

① $n=1 \quad (a_1 b_1)^2 = a_1^2 b_1^2$

② $P(n) \to P(n+1)$

$A = \sum_{i=1}^{n} a_i^2$

$B = \sum_{i=1}^{n} a_i b_i$

$C = \sum_{i=1}^{n} b_i^2$

Prove $(B + a_{n+1}b_{n+1})^2 \le (A + a_{n+1}^2)(C + b_{n+1}^2)$

$\iff (AC - B^2) + A b_{n+1}^2 + C a_{n+1}^2 - 2B a_{n+1} b_{n+1} \ge 0$

Since $AC \ge B^2 \iff (AC - B^2) + (\sqrt{A} b_{n+1} - \sqrt{C} a_{n+1})^2 \ge 0$

[e.g.3] strong induction

p_i the i-th prime number prove $p_n < 2^{2^n}$ for an arbitrary $n \in \mathbb{N}^+$

proof: ① $P_1 = 2 < 2^{2^1} = 4$

② $P(n) \to P(n+1)$

let $N = \prod_{i=1}^{n} p_i + 1$ (N is a prime or $\exists q. q \mid N$ s.t $q \ge p_{n+1}$)

$\Rightarrow p_{n+1} \le q \le N$

$\Rightarrow p_{n+1} \le \prod_{i=1}^{n} p_i + 1 < 2^{2^1 + \cdots + 2^n} + 1 = 2^{2^{n+1}-2} + 1 < 2 \cdot 2^{2^{n+1}-2} = 2^{2^{n+1}-1} < 2^{2^{n+1}}$

$\Rightarrow p_n < 2^{2^n} \quad \forall n \in \mathbb{N}^+$

• Well ordering principle 良序原理

Every nonempty set of nonnegative integers has a smallest element.

[e.g.] prove $\sqrt{2}$ is irrational

Proof: Suppose $\sqrt{2}=\frac{m}{n}$ $(\gcd(m,n)=1)$, a rational number (By WOP, $\exists\, |m|$ s.t. $\sqrt{2}=\frac{m}{n}$)

$\Rightarrow 2=\frac{m^2}{n^2} \Rightarrow m^2=2n^2$ even number $\quad m=2k \Rightarrow 2k^2=n^2$

$\Rightarrow n$ even number

$\Rightarrow$ contradict with $\gcd(m,n)=1$

$\Rightarrow \sqrt{2}$ is irrational

• recursion

1. catalan number $\quad r_n=\sum_{k=1}^{n} r_{k-1}r_{n-k}=\frac{1}{n+1}\binom{2n}{n}$

2. generating functions $\quad f(x)=\sum_{i=0}^{\infty} a_i x^i$

$$\begin{cases} (\underbrace{0,0,\cdots,0}_{k},f_0,f_1,f_2) \Leftrightarrow x^k f(x) \\ (f_0,f_1,\cdots) \Leftrightarrow f(x) \\ (f_1,2f_2,\cdots) \Leftrightarrow f'(x) \end{cases}$$

$(1,1,\cdots,1) \Leftrightarrow \frac{1}{1-x}$

$(1,2,3,\cdots) \Leftrightarrow \left(\frac{1}{1-x}\right)'=\frac{1}{(1-x)^2}$

$(0,1,4,9,\cdots) \Leftrightarrow \frac{x(1+x)}{(1-x)^3}$

[e.g.1] $\quad f_0=0,\ f_1=1,\ f_n=f_{n-1}+f_{n-2}$

$f(x)=f_0+f_1x+\cdots=0+x+(f_0+f_1)x^2+(f_1+f_2)x^3+\cdots$

$$\begin{array}{rcccccl} & \langle 0 & 1 & 0 & 0 & \cdots \rangle & \Leftrightarrow x \\ & \langle 0 & f_0 & f_1 & f_2 & \cdots \rangle & \Leftrightarrow xf(x) \\ + & \langle 0 & 0 & f_0 & f_1 & \cdots \rangle & \Leftrightarrow x^2f(x) \\ \hline & \langle 0 & f_0+1 & f_0+f_1 & f_1+f_2 & \cdots \rangle & \Leftrightarrow f(x) \end{array}$$

$f(x)=x+xf(x)+x^2f(x)$

$f(x)=\frac{x}{1-x-x^2}=\frac{1}{\sqrt{5}}\left(\frac{1}{1-\alpha_1x}-\frac{1}{1-\alpha_2x}\right) \quad \alpha_1=\frac{1+\sqrt{5}}{2} \quad \alpha_2=\frac{1-\sqrt{5}}{2}$

↓ taylor series

$f(x)=\frac{1}{\sqrt{5}}\sum_{n=0}^{\infty} f_n x^n=(\alpha_1^n-\alpha_2^n)x^n$

$\Rightarrow f_n=\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n-\left(\frac{1-\sqrt{5}}{2}\right)^n\right]$

[e.g.2] use GF solving Catalan $C(x) = C_0 + x\,C(x)^2$ $\left[C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i},\ C_0 = 1 \right.$

$$C(x) = \sum_{n=0}^{\infty} C_n x^n$$

$$C_n = \sum_{i=0}^{n-1} C_i C_{n-i-1} (n \ge 1) \Rightarrow \sum_{n=1}^{\infty} C_n x^n = \sum_{n=1}^{\infty} \left(\sum_{i=0}^{n-1} C_i C_{n-i-1} \right) x^n$$

$$= x \sum_{m=0}^{\infty} \sum_{i=0}^{m} C_i C_{m-i} x^m \quad (m = n-1)$$

$$= x\, C(x)^2 \quad [\text{卷积}]$$

$$\sum_{n=1}^{\infty} C_n x^n = C(x) - C_0 = C(x) - 1$$

$$\Rightarrow C(x) = x\, C(x)^2 + 1$$

$$C(x) = \frac{1-\sqrt{1-4x}}{2x} = \frac{1 - \sum_{i=0}^{\infty} \binom{\frac{1}{2}}{i} (-4x)^i}{2x} = \sum_{i=1}^{\infty} \frac{[2(i-1)]!}{2^{i-1} (i-1)!(i-1)!\, i} (2x)^{i-1}$$

$$= \sum_{i=0}^{\infty} \frac{(2i)!}{(i!)^2 (i+1)} x^i = \sum_{i=0}^{\infty} \frac{1}{i+1} \binom{2i}{i} x^i$$

$$\Rightarrow r_n = \frac{1}{n+1} \binom{2n}{n} \quad (r_0 = 1)$$

3. Second order Recurrence

$a_k = A a_{k-1} + B a_{k-2} \quad (B \ne 0)$

Assume $A_n = r^n$

$\Rightarrow r^n = A r^{n-1} + B r^{n-2}$

$\Rightarrow r^2 - Ar - B = 0$ [特征方程]

$$\begin{cases} a_0 = C r_1^0 + D r_2^0 = C + D \\ a_1 = C r_1 + D r_2 \end{cases}$$

$$\Rightarrow a_n = C r_1^n + D r_2^n \quad \left(C = \frac{a_1 - a_0 r_2}{r_1 - r_2},\ D = \frac{a_0 r_1 - a_1}{r_1 - r_2} \right)$$

* $A a_n = B a_{n-1} + C a_{n-2}$

$A(a_n - q a_{n-1}) = k(a_{n-1} - q a_{n-2})$

$A a_n - (Aq + k) a_{n-1} + kq a_{n-2} = 0$

$$\begin{cases} -(Aq + k) = B \\ kq = -C \end{cases} \Rightarrow Aq - \frac{C}{q} = B \Leftrightarrow Aq^2 - Bq - C = 0$$

[特征方程]

$$\left[a_n = \alpha (q_1)^n + \alpha_2 (q_2)^n \right]$$

4. gcd

prove $\gcd(a,b) = \gcd(b, a \bmod b)$ $(a < b)$ $a = qb + r$

① $a \bmod b = 0$

$a = qb \Rightarrow \gcd(a,b) = b = \gcd(b,0)$

② $a \bmod b = r > 0$

1° let $b = k_1 d,\ r = k_2 d$
$a = qb + r = (qk_1 + k_2)d$ $\Rightarrow d$ is a common divisor of $a.b$

2° let $a = k_3 d,\ b = k_1 d$
$r = (k_3 - qk_1)d$ $\Rightarrow d$ is a common divisor of b,r

$\{$common factors of $a.b\} = \{$common factors of $b.r\}$

$\Rightarrow \gcd(a,b) = \gcd(b,r)$

$*\ a = bq + r \geq b + r \geq 2r$ $\gcd(a,b) = \gcd(b,r)$ $\boxed{r < \frac{a}{2}}$

if $b \in [2^d, 2^{d+1})$ $\Rightarrow \frac{b}{2^k} \leq 1$ $k = \log_2 b$

5. spc smallest positive integer linear combination

$d = sa + tb$ $(s,t \in \mathbb{Z}) \Rightarrow d$ is an integer linear combination

$spc(a,b) = \min\{sa + tb > 0 \mid s,t \in \mathbb{Z}\}$

$*\gcd(a,b) = spc(a,b)$

① $\gcd \leq spc$

$a = k_1 d,\ b = k_2 d$ $sa + tb = d(sk_1 + tk_2) \geq d \Rightarrow d \mid (sa + tb)$

$\Rightarrow \gcd(a,b) \leq spc(a,b)$

② $spc \leq \gcd$

$a = q\,spc(a,b) + r$ $r \in [0, spc(a,b)]$

$r = a - q(sa + tb) = (1 - qs)a + (-qt)b$

if $r > 0$, $r < spc(a,b) \xleftrightarrow{\text{contradiction}} r$ is an integer linear combination

$\Rightarrow$ only possible when $r = 0$

$\Rightarrow spc(a,b) \mid a$

similarly. $spc(a,b) \mid b$

$\Rightarrow spc(a,b) \leq \gcd(a,b)$

$\Rightarrow spc(a,b) = \gcd(a,b)$

* $\gcd(a,b)=1,\ \gcd(a,c)=1 \Rightarrow \gcd(a,bc)=1$

Proof:

$spc(a,b)=\gcd(a,b)=sa+tb=1$

$spc(a,c)=\gcd(a,c)=ua+vc=1$

$(sa+tb)(ua+vc)=(sau+svc+tbu)a+(tv)bc=1$

$\Rightarrow spc(a,bc)=1>0$

$\Rightarrow \gcd(a,bc)=1$

* p is a prime $\quad p\mid ab \Rightarrow p\mid a$ or $p\mid b$

Proof: assume $p\nmid a$, $\gcd(p,a)=1 \Rightarrow sa+tp=1$

$\Rightarrow sab+tpb=b$

Since $p\mid ab,\ p\mid p \Rightarrow p\mid b$ $\quad$ [$p\mid a$ and $p\mid b \Rightarrow p\mid a$ or $p\mid b$ 显然成立]

$\Rightarrow$ p is a prime and $p \mid \prod_{i=1}^{m} a_i \Rightarrow p\mid a_i$ for some i

6. exgcd $\quad \gcd(a,b)=sa+tb$

[e.g.] $\gcd(259,70)=\boxed{7}$

$259=3\times70+49$

$70=1\times49+21$

$49=2\times21+7$

$21=3\times7+0$

$49=259-3\times70=a-3b$

$21=70-1\times49=b-(a-3b)=-a+4b$

$7=3a-11b=\boxed{3\times259+(-11)\times70}$

general solution to solve $ax+by=c$

① $\gcd(a,b)\nmid c$ $\quad$ Impossible [$d=\gcd(a,b)\Rightarrow d\mid ax+by$]

② $\gcd(a,b)\mid c$ $\quad$ let $a<b$ $\quad$ exist $ax-ny=c$ $\quad$ $(n=-b)$

例如问题 $\quad ax \bmod y=c$

- **Modular Arithmetic**

☆ $m|n$, $a \equiv b \pmod n \Rightarrow a \equiv b \pmod m$

$n | a-b \Rightarrow q n = a-b \Rightarrow q t m = a-b$

$\Rightarrow m | a-b \Rightarrow a \equiv b \pmod m$

1. $a \equiv b \pmod n$ iff $n | (a-b)$

   $[a]_b = \{a+kb : k \in \mathbb{Z}\}$

2. if $\gcd(k,n)=1$, then have k' s.t. $k \cdot k' \equiv 1 \pmod n$

① $\gcd(k,n) = spc(k,n) = 1 \Rightarrow sk + tn = 1 \Rightarrow tn = 1 - sk$

$\Rightarrow 1 - sk \equiv 0 \pmod n$

$\Rightarrow sk \equiv 1 \pmod n$ $\quad k' = s$

② k' is unique $\pmod n$

Assume exist k_1, k_2 $(k_1 \neq k_2)$ $\quad k k_1 \equiv k k_2 \equiv 1 \pmod n \Rightarrow k(k_1 - k_2) \equiv 0 \pmod n$

$k k_1 (k_1 - k_2) \equiv k_1 - k_2 \equiv 0 \pmod n \Rightarrow k_1 \equiv k_2 \pmod n$

* if $i \cdot k \equiv j \cdot k \pmod n$ and $\gcd(k,n) = 1$

$i \cdot k \equiv j \cdot k \pmod n \Rightarrow i \cdot k \cdot k' \equiv j \cdot k \cdot k' \pmod n \Rightarrow i \equiv j \pmod n$

3. - **Fermat's Little Theorem**

let p be a prime and $\gcd(k,p)=1$, then $k^{p-1} \equiv 1 \pmod p$

proof

$(p-1)! \equiv (k \bmod p)(2k \bmod p) \cdots ((p-1)k \bmod p) \pmod p$

$\equiv k^{p-1}(p-1)! \pmod p$

p is a prime $1, \cdots p-1$ are coprime with $p \Rightarrow k^{p-1} \equiv 1 \pmod p$

- **Wilson's Theorem** $\quad p$ is a prime $\Longleftrightarrow (p-1)! \equiv -1 \pmod p$

proof: ① prime $\Rightarrow (p-1)! \equiv -1 \pmod p$ $\quad$ 每个数均有唯一逆元 $\quad$ 凑对

$(p-1)! \equiv 1 \cdot p-1 \cdot (a_1 b_1) \cdots (a_{\frac{p-3}{2}} b_{\frac{p-3}{2}}) \pmod p$ $\quad (a_i b_i \equiv 1 \pmod p)$

$\equiv 1 \cdot (p-1) \cdot 1 \cdots 1 \pmod p$

$\equiv -1 \pmod p$

② $(p-1)! \equiv -1 \pmod p \Rightarrow$ prime

Assume p is not a prime

$\exists q$, s.t $q | p$ (q is a prime) $\Rightarrow q \leq p-1 \Rightarrow q | (p-1)! \Rightarrow (p-1)! \equiv 0 \pmod q$

$(p-1)! \equiv -1 \pmod p \Rightarrow (p-1)! \equiv -1 \pmod q$

$\Rightarrow$ Contradiction $\Rightarrow p$ is a prime

4. Chinese Remainder Theorem

<1> $ax \equiv b \pmod{n}$ $\quad g = \gcd(a, n)$

① $g = 1$ $\quad x \equiv a^{-1} b \pmod{n}$

② $g > 1$ $\quad$ if $g \mid b$, $ax \equiv b \pmod{n} \Leftrightarrow ax = b + tn \Leftrightarrow a'gx = b'g + tn'g$

$\Leftrightarrow a'x = b' + tn'$

$\Leftrightarrow a'x \equiv b' \pmod{n'}$ $\quad \gcd(a', n') = 1$

otherwise [Assume $\exists t$]

$b = (a'x - n't)g$ [contradiction with $g \nmid b$] $\Rightarrow$ no solution

$\Rightarrow$ has a solution iff $\gcd(a, n) \mid b$

<2> $\begin{cases} x \equiv a_1 \pmod{n_1} \\ \vdots \\ x \equiv a_k \pmod{n_k} \end{cases}$ has a unique solution modulo $n = \prod_{i=1}^{k} n_i$ if $n_1, \cdots n_k$ be mutually coprime.

①

let $N_i = \dfrac{\prod_{j=1}^{k} n_j}{n_i}$, $\gcd(N_i, n_i) = 1 \Rightarrow \exists x_i$ s.t. $N_i x_i \equiv 1 \pmod{n_i}$

let $x = \sum_{i=1}^{k} N_i(x_i a_i)$. since $n_i \mid N_j \mid_{j = \{1, 2, \cdots k\} \setminus i}$

$\Rightarrow x \equiv N_i(x_i a_i) \equiv a_i \pmod{n_i}$

② uniqueness Assume x, y be solutions $\quad \forall i$ $x - y \equiv 0 \pmod{n_i} \Rightarrow n_i \mid x - y$

$n_1, \cdots n_k$ are mutually coprime $\Rightarrow n_1 \cdots n_k \mid x - y \Rightarrow x \equiv y \pmod{n_1 \cdots n_k}$

<3> $n_1 \cdots n_k$ 不互质的情况 $\quad$ 一个个式子代入解决 $N = \operatorname{lcm}(n_1, \cdots n_k)$

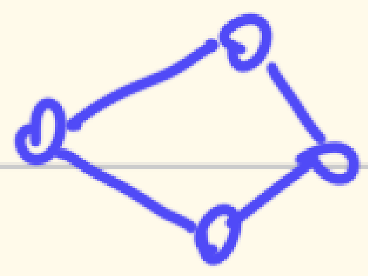
· graph

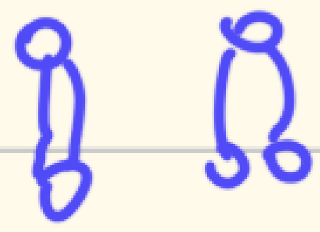
1. $2|E| = \sum_{v \in V} \deg(v)$

2. Graph Isomorphism G_1 is isomorphic to G_2 means $\exists$ one-to-one mapping $f: V_1 \to V_2$ $u-v$ in E_1 iff $f(u)-f(v)$ in E_2

* isomorphic $\xrightarrow{\quad}$ have the same degree sequences ($\not\leftarrow$)

[e.g.]





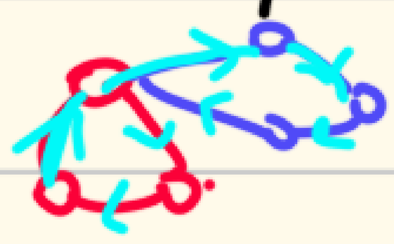
[2,2,2,2]

3. Euler Theorem [欧拉路径 每条边经过一次]

① A connected graph has an Eulerian path iff it has 0 or 2 vertices with odd degrees.

② A connected graph has an Eulerian Cycle iff every vertex is of even degree.

*可拆分成若干环. 每个环、先走一半.再走另半



*prove 2个奇点 → 存在欧拉路径

Proof: Assume u and v are the two vertices with odd degrees.

Add u-v to form G' ⇒ G' all have even degrees ⇒ have Eulerian Cycle

remove u-v ⇒ start at u. end at v (a path that uses each edge exactly once)

4. direct graph

- weakly connected 忽略方向后连通
- strongly connected 任意两点可达
- directed cycle 强连图，每个点出入度均为 1

5. Stable match $O(n^2)$

.*贪心找最优 . 一定存在一组解

· Proposers will get the best possible matches almong all possible stable matchings.

Being proposed will get the worst possible matches almong all possible stable matchings.

①. 有效伴侣 $\exists$ stable match M s.t. $(B, G) \in M$

B: the first boy rejected by valid parter G (G 为最佳有效伴侣)

$\Rightarrow B' >_G B$ $\quad M' \Rightarrow (B', G) \in M'$. $(B, G') \in M'$

if B' prefers G' rather than G: $B' \to G'$ first, then G

① G' accepted $(B', G') \in M'$ rather than $(B', G) \in M$ contradiction

② rejected contradict with B is the first boy being rejected.

↑说明B'先被G'拒绝才能与G配对，而B又被G拒绝

$\Rightarrow G \succ_{B'} G'$

$\Rightarrow \begin{cases} B' \succ_G B \\ G \succ_{B'} G' \end{cases} \Rightarrow$ unstable

② $(B, G) \in M$ M': $(B', G) \in M'$ $(B, G') \in M'$

if exist B': $B \succ_G B'$ and (B', G) is a valid partner

in ① we can know $(B, G) \in M \Rightarrow G \succ_B G'$

$\begin{cases} B \succ_G B' \\ G \succ_B G' \end{cases} \Rightarrow$ unstable

6. bipartite Graph

a subset of edges s.t. $\forall v \in V$, $\deg(v) = 1$

↓

matching $\begin{cases} \text{perfect 覆盖所有点} \\ \text{maximum 边最多} \\ \text{bipartite} \\ \text{stable} \end{cases}$

• Hall's Theorem

A bipartite graph $G = (V, W, E)$ has a perfect matching iff $|N(S)| \geq |S|$ for each subset S of V and W. [$N(S) = \{v \mid v$ is a neigber of some vertex in $S\}$]

[e.g.] Show that a tree has at most one perfect matching.

Assume T has a different match M_1, M_2, let $T' = (V, M_1 \Delta M_2)$

$T' = (V, M_1 \Delta M_2)$ 对称差 $\forall v \in V$, the associated edge,

① $(u, v) \in M_1 \cap M_2 \Rightarrow (u, v) \notin M_1 \Delta M_2$, v在T'中 $\deg(v) = 0$

② $(u, v) \notin M_1 \cap M_2 \Rightarrow (u, v) \in M_1 \Delta M_2$. v在T'中 $\deg(v) = 2$

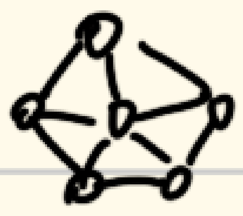
$\Rightarrow \deg_{T'}(v) \in \{0, 2\}$

$\Rightarrow M_1 \neq M_2$ $M_1 \Delta M_2 \neq \emptyset$ $\Rightarrow$ T'成环

7. $\chi(G)$ Chromatic number = Minimum number of colors to color G

① Simple circle $\chi(C_{even})=2$ $\chi(C_{odd})=3$ ② complete graphs $\chi(K_n)=n$

③ wheels $\chi(W_{odd})=4$ $\chi(W_{even})=3$

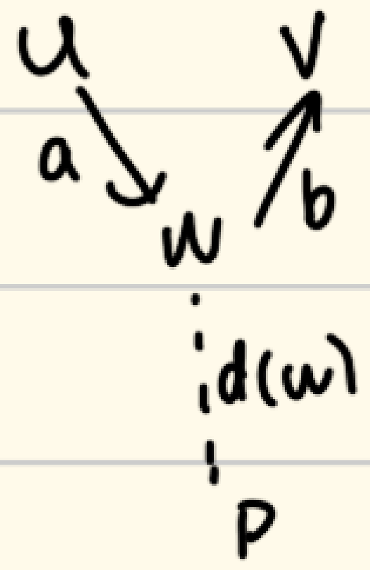


- A graph is 2-colorable iff it is bipartite.

① $A \to B \to A$ even-length path can return to the same partition

② p [red] bfs 奇偶染色

Assume (u,v) same color $\Rightarrow d(u) \bmod 2 = d(v) \bmod 2$



$u \Rightarrow w \Rightarrow v$ $d=a+b$

$d(u)=d(w)+a$ $d(v)=d(w)+b$

$\Rightarrow a \bmod 2 = b \bmod 2$ $u \Rightarrow w \Rightarrow v \Rightarrow u$ $d'=a+b+1$ odd

$\Rightarrow (u,v)$ same color D.N.E

$\omega G \le \chi(G) \le \max_{v\in V}(\deg(v))+1$

- $\omega(G)$ the largest size of a complete subgraph that G contains $\Rightarrow \chi(G) \ge \omega(G)$

[interval graph $\omega(G)=\chi(G)$ $\leftarrow$ 最大度为d 则 $\chi(G) \le d+1$]

8. planar graphs [there is a way to draw the graph on a plane without the edges crossing]

connected planar graph n vertices, m edges, f faces $\boxed{n-m+f=2}$
$V-E+F$

- k connected components

$$n-m+f=\sum n_i-\sum m_i+\sum f_i-(k-1)=k+1$$

☆简单平面图 $\sum \deg(\text{face})=2E\ge 3F$

- 6 coloring

* G is a simple planar graph $(n\ge 3)$, then $m\le 3n-6$

$$2m=\sum_{i=1}^{f} F_i \text{ (第i个面的边数)} \ge 3f$$ [each face has at least 3 edges]

$$n-m+f=2 \Rightarrow n-m+\frac{2}{3}m\ge 2 \Rightarrow m\le 3n-6$$

$\to$ every vertex's degree ≤ 5 [assume $\ge 6 \Rightarrow m \ge \frac{6n}{2}=3n$]

[e.g.] $G=(V,E)$ be a graph that every two odd cycles in G has at least one common vertex. Show G is 5-colorable.

① no odd cycles bipartite graph $\Rightarrow$ 2-colorable

② let $\{v_0,\cdots,v_n\}$ be an odd cycle C, $G'=\{V',E'\}$ $V'=V\setminus\{v_0,\cdots v_n\}$

$\Rightarrow G'$ is bipartite graph [否则两奇环无共点] $\Rightarrow$ 2-colorable

C is an odd cycle $\Rightarrow$ 3-colorable

$\Rightarrow$ G is 5-colorable

8. combinatorics

- inclusion-exclusion (n sets)

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i| - \sum_{1\le i<j\le n} |A_i \cap A_j| \cdots + (-1)^{n+1} |A_1 \cap A_2 \cap \cdots \cap A_n|$$

$$\binom{k}{1} - \binom{k}{2} + \cdots + (-1)^{k+1}\binom{k}{k} = 1$$

错位排列: let $h(n,k)$ 表示 n个里面k位是对的的方案数

$$h(n,k) = \binom{n}{k}\cdot(n-k)!$$

$$f(n) = n! - \sum_{i=1}^{n}\binom{n}{i}\cdot(n-i)!\,(-1)^{i-1} = n!\sum_{i=0}^{n}\frac{(-1)^n}{n!}$$

- 欧拉函数 Euler's Totient Function

$$\varphi(n) = n - \sum_i \frac{n}{p_i} + \sum_{i<j}\frac{n}{p_i p_j} + \cdots + (-1)^r\frac{1}{p_1\cdots p_r} = n\prod_{i=1}^{r}\left(1-\frac{1}{p_i}\right)$$

9. mapping

if $f(x_1)=f(x_2)$, $x_1=x_2$

if $x_1\ne x_2$, $f(x_1)\ne f(x_2)$

- injection 单射 no two inputs have the same output.
- surjection 满射 every outputs is possible
- bijection 双射 $|A|=|B|$ [一一对应]

对于有限集
$\|A\|\le\|B\|$
$\|A\|\ge\|B\|$

- If f is a bijection from A to B, then $|A|=|B|$

[e.g.] prove $f(x)=2x$ is bijective

① injective suppose $f(x_1)=f(x_2)$ $2x_1=2x_2 \Rightarrow x_1=x_2$

② surjective $\forall y\in\mathbb{R}$ $x=\frac{y}{2}\in\mathbb{R}$ s.t. $f(x)=2x=2\cdot\frac{y}{2}=y$

- map counting

[e.g.] $x_1+x_2=n$, $x_1<a_1$, $x_2<a_2$ $(x_1,x_2\ge 0)$

① $A_1=\{(x_1,x_2)\mid x_1+x_2=n \wedge x_1\ge 0, x_2\ge 0\}$

$|A_1| = \binom{n+2-1}{2-1} = n+1$

② $A_2=\{(x_1,x_2)\mid x_1+x_2=n,\ x_1\ge a_1,\ x_2\ge 0\}$

$|A_2| = \binom{n-a_1+2-1}{2-1} = n-a_1+1$

③ $A_3=\{(x_1,x_2)\mid x_1+x_2=n,\ x_1\ge a_1, x_2\ge a_2\}$

$|A_3| = \binom{n-a_1-a_2+2-1}{2-1} = n-a_1-a_2+1$

$\Rightarrow$ answer: $(n+1) - \max(0, n-a_1+1) - \max(0, n-a_2+1) + \max(0, n-a_1-a_2+1)$

[e.g.2] the number of cnt.

```
for i:=1 to n do
  for j:=1 to i do
    for k:=1 to j do
      cnt +:=1
```

① $x_1=k-1,\ x_2=j-k,\ x_3=i-j,\ x_4=n-i\ \ (x_i\geq 0)$

$\Rightarrow x_1+x_2+x_3+x_4=n-1 \qquad \binom{n-1+4-1}{4-1}=\binom{n+2}{3}$

② x_i 表示 i 被选 x_i 次

$\Rightarrow x_1+\cdots+x_n=r=3\ \ (x_i\geq 0) \qquad \binom{r+n-1}{r}$

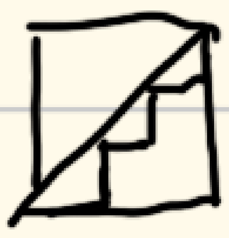
$\Rightarrow \binom{n+3-1}{n-1}=\binom{n+2}{n-1}=\binom{n+2}{3}$

[e.g.3] n本书排成一排. 选k本. 要求不相邻

$\boxed{x_1}\ 1\ \boxed{x_2}\ 2\ \boxed{x_3}\ 3\ \cdots\ k\ \boxed{x_{k+1}}$

$x_1+x_2+\cdots+x_{k+1}=n-k \quad (x_1\geq 0,\ x_2\geq 1,\cdots x_k\geq 1,\ x_{k+1}\geq 0)$

$\Rightarrow x_1+\cdots+x_{k+1}=n-k-(k-1)=n-2k+1\ (x_i\geq 0) \Rightarrow \binom{(n-2k+1)+(k+1)-1}{k+1-1}=\binom{n-k+1}{k}$

- division rule $\quad A\to B$ is k-to-1, $|A|=k|B|$
- Catalan number  $r_n=\frac{1}{n+1}\binom{2n}{n}=\binom{2n}{n}-\binom{2n}{n-1}$

|A| $(0,0)\Rightarrow(n,n)$ 单调路径 (要求非 Low-right. i.e. 存在某点 $y>x$)

|B| 反射原理得到的(关于 $y=x+1$ 对称) $(0,0)\Rightarrow(n-1,n+1)$ 单调路径

① injection $|A|\leq|B|$ different paths flip to different paths

② surjection $|A|\geq|B|$ every path in A must "cross" the diagonal at least once

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